Non-Gaussian, Mixed Continuous-Variable Entangled States

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Abstract

We study a class of mixed non-Gaussian entangled states that, whilst closely related to Gaussian entangled states, none-the-less exhibit distinct properties previously only associated with more exotic, pure non-Gaussian states.

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I. INTRODUCTION

States with non-Gaussian quadrature probability distributions are of high interest in quantum optics. Pure versions of such states are difficult to generate experimentally as they require either very high non-linearity [1] or difficult conditional preparation [2]. If we include multi-mode and mixed non-Gaussian states we open up a huge arena that is in general not fully understood. In contrast quantum states of light which have Gaussian quadrature distributions are well understood [3]. This now includes their entanglement properties [4]. Numerous quantum information applications for Gaussian states have been proposed [5] and demonstrated [6], [7], [8], [9].

Here we propose an entangled state which is non-Gaussian but could be considered one step away from a Gaussian state. The state is a random mixture of two distinct Gaussian states. This is quite clearly non-Gaussian as adding two Gaussian distributions with different means or different variances will result in either a "two peaked" distribution or a distribution where the higher order moments are not what is expected from a Gaussian distribuiton. So-called "proper mixtures" of this kind, i.e. where the mixture is created by introducing a classical random variable, have been studied extensively in the discrete variable domain of photon counting experiments [10], [11]. We show that our proposed continuous variable mixed states exhibit interesting properties that have previously only been observed in the context of models of more exotic, pure non-Gaussian states. Here the effects are found in a more accessible scenario.

We begin in section II by introducing the measures of entanglement that we will use for our analysis. Section III will discuss the entanglement properties of Gaussian states and section IV will extend this to our mixed states. In section VI we discuss two examples: continuous variable quantum teleportation [6] and; the entanglement cloner attack in coherent state quantum key distribution [7]. Finally in section VIII we will conclude.

II. ENTANGLEMENT MEASURES

In this paper we will require some way to compare entangled states. The measures used here are the so called 'negativity' [12] and the 'inseparability criterion' [4]. These measures contain most of the properties that one would expect from a measure of entanglement. We

choose only these two here as they are comparitively easy to compute in the continouous variable regime.

A. Negativity

We will first introduce the negativity [12]. This entanglement measure is the sum of the negative eigenvalues of the partial transpose of the density operator [13]. In [12] the properties of this measure are studied and it is shown that this value can be computed by evaluating

$$\mathscr{N}(\hat{\rho}) = \frac{\|\hat{\rho}^{T_A}\|_1 - 1}{2}, \tag{1}$$

where $\|\cdot\|_1$ is the usual trace norm (i.e. $tr\sqrt{\hat{A}^\dagger \hat{A}}$) and $\hat{\rho}^{T_A}$ is the partial transpose of the density operator [13].

B. Inseparability criterion

The 'inseparability criterion' [4] using the classic EPR type observables [14] provides a value which can be used as a sufficient condition for inseparability. This is useful as it gives a way of knowing if a state is entangled by calculating values from averages of obserable quantities. The inseparability criterion also has an operational meaning in CV teleportation. There is a one-to-one correspondance between it and the fidelity of teleportation if the entanglement resource has a symmetric Gaussian Wigner function. This is an important point for this work, as later we will use this fact to argue that the inseparability criterion is a good quantity to use to compare continuous variable entangled states.

If we choose our energy scale so that shot noise has unit variance then the inseparability criterion can be written

$$\frac{1}{4}\langle\left[\Delta(\hat{X}_1^+ + \hat{X}_2^+)\right]^2\rangle + \frac{1}{4}\langle\left[\Delta(\hat{X}_1^- - \hat{X}_2^-)\right]^2\rangle < 1 \tag{2}$$

where the subscripts 1 and 2 represent the two modes in which the entanglement exists and $\hat{X}_j^+ = \hat{a}_j + \hat{a}_j^\dagger$, $\hat{X}_j^+ = i(\hat{a}_j - \hat{a}_j^\dagger)$ are the in-phase and out-of-phase quadrature amplitudes respectively. Also $\hat{a}_j$ and $\hat{a}_j^\dagger$ are the anhillation and creation operators respectively for the modes. Note that this expression is only valid when considering entanglement that has symmetric correlations, both between the modes and between their quadartures (technically,

the entanglement is said to be in the standard form [4]). The expression on the left hand side of equation 2 we shall call the 'inseparability criterion number' or just inseparablity criterion when the meaning obviously implies a numerical value as opposed to an inequality.

III. TWO-MODE GAUSSIAN STATES

The Gaussian states we will use as ingredients for our non-Gaussian mixed states are the two mode squeezed vacua. We can write

$$\hat{S}(r)\left|0,0\right\rangle = e^{r(\hat{a}_1\hat{a}_2-\hat{a}_1^\dagger\hat{a}_2^\dagger)}\left|0,0\right\rangle \tag{3}$$

where $\hat{S}(r)$ is the 'squeeze' operator with r as a parameter (here assumed real) which determines the amount by which the state is squeezed (squeeze parameter). This state is Gaussian and hence is completely characterized by a vector of means and a covariance matrix. A two mode state requires four phase space coordinates and we will choose to write them in a vector of the form (x_1, p_1, x_2, p_2). Squeezed vaccua are centered around the origin of phase space, so the vector of means is the zero vector. The covariance matrix for the two mode squeezed vacuum as defined in Eq. 3 is

$$\begin{pmatrix} \cosh(2r) & 0 & -\sinh(2r) & 0 \\ 0 & \cosh(2r) & 0 & \sinh(2r) \\ -\sinh(2r) & 0 & \cosh(2r) & 0 \\ 0 & \sinh(2r) & 0 & \cosh(2r) \end{pmatrix}. \tag{4}$$

Given these variances we are now in a position to evaluate the inseparability criterion for this state. It is

$$\cosh(2r) - \sinh(2r) = e^{-2r}. \tag{5}$$

Note that this value is less than one and hence is entangled for all non-zero, positive r.

The two mode squeezed vaccum when written in the Fock basis is

$$\hat{S}(r)\left|0,0\right\rangle = \sum_{n=0}^{\infty}(\tanh r)^n \left|n,n\right\rangle. \tag{6}$$

Using this basis, one can show that the negativity of this state is

$$\mathcal{N}(\hat{\gamma}(r)) = \frac{\tanh r}{1-\tanh r} = \frac{e^{2r}-1}{2}. \tag{7}$$

Using this state as the entanglement resource to teleport an unknown coherent state using continuous variable teleportation [15] (with unity gain) results in an output state which when compared with the input state has a fidelity of

$$\mathscr{F}_{\hat{S}(r)|0,0\rangle} = \frac{1}{1+e^{-2r}}. \tag{8}$$

Note here that the fidelity is in one-to-one correspondance with the inseparability criterion. This is only true for symmetric Gaussian states.

IV. MIXTURES OF GAUSSIAN STATES

Consider the mixed state which is a random mixture of vaccum state and two mode squeezed state, i.e.

$$\hat{\rho}_{rm1} = p\hat{\gamma}(r) + (1-p)\hat{\gamma}(0) \tag{9}$$

where $\hat{\gamma}(r)$ represents the density operator associated with a Gaussian two mode squeezed vaccum with squeeze parameter r and hence p is the probability that the state is actually the *squeezed* vaccum. Note here that $\hat{\gamma}(0)$ is the un-squeezed vaccum and hence is the usual vaccum state.

To calculate the inseparability criterion for this state involves taking the expectation value of given operators (see Eq. (2)). Expectation values are calculated in quantum mechanics by calculating $\langle\hat{A}\rangle = Tr\left\{\hat{\rho}\hat{A}\right\}$. This equation is linear in both $\hat{A}$ and $\hat{\rho}$. So the inseparability criterion reduces to evaluating the weighted sum of the inseparability criterions of the two states which make up the mixture. That is

$$pe^{-2r} + (1-p). \tag{10}$$

The negativity of the mixture is given by

$$p(\frac{e^{2r}-1}{2}). \tag{11}$$

V. COMPARISON

One way to compare the Gaussian state with the non-Gaussian state would be to compare the two states with the same r (squeezing). This type of comparison was made by Mista et al in the context of similar, thermal/squeezed state mixtures [16]. It is clear that with such

a criteria the non-Gaussian state will always be inferior to the Gaussian state. However, we believe that this is not a fair comparison of the two states.

Our method for comparing these two states is by way of the inseparability criterion. We take this stance as the inseparability criterion is generally accepted as a measure of the "strength" of entanglement for Gaussian states. Operationally, as we have seen, it is in one-to-one correspondence with the fidelity of CV teleportation of coherent states and is easily measured via the second order moments. We are interested in ways in which our non-Gaussian state may behave differently from Gaussian states, thus it makes sense to compare states with the same "strength" of entanglement, i.e. the same inseparability criterion.

If we take a mixture of two Gaussian states as per Eq. (9) then a pure two mode squeezed state would have the same inseparability criterion when

$$e^{-2s} = pe^{-2r} + (1-p) = I \tag{12}$$

where s is the squeeze parameter for the pure state. Here we are defining I to be the common inseparability criterion. As $0 < e^{-2r} \leq 1$ and $0 \leq p \leq 1$ we find that $1 - p < I \leq 1$. It is important to note here that as the squeezing increases, I decreases. So the lowest possible I corresponds to the maximum squeezing.

A. Negativity

The negativity of the non-Gaussian state of Eq. (9) can be expressed in terms of the I using Eq. (12) as

$$\mathscr{N}(\hat{\rho}_{rm1}) = \frac{p^2}{2(I+p-1)} - \frac{p}{2}. \tag{13}$$

Similarly the negativity of the Gaussian state can be written

$$\mathscr{N}(\hat{\gamma}(s)) = \frac{I^{-1}-1}{2}. \tag{14}$$

The Eqs. (13) and (14) are plotted for comparison in Fig. 1. It would appear that the negativity of the non-Gaussian state is always higher than that of a Gaussian state with the same inseparability criterion. To prove this we consider the difference between the mixed state negativity (Eq. (13)) and the pure state negativity (Eq. (14)):

$$\frac{p^2}{2(I+p-1)} - \frac{p}{2} - \frac{I^{-1}-1}{2} = \frac{(I-1)^2(p-1)}{2I(I-1+p)}, \tag{15}$$

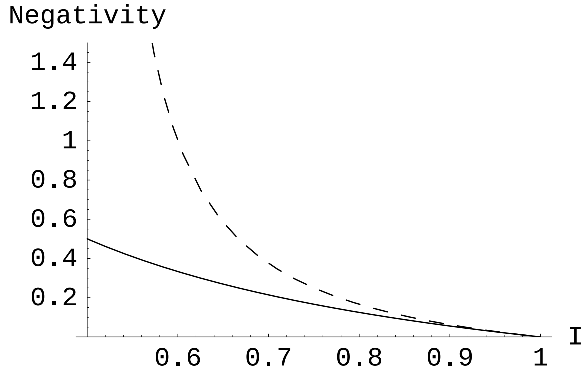

FIG. 1: A plot of the negativity of the mixed state as per Eq. (13) (dashed) and the equivalent pure state with the same inseparability criterion as per Eq. (14) (solid). This plot is in the same style as Fig. 2.

which is always a positive quantity as $0 \leq p \leq 1$ and $1 - p < I \leq 1$.

We conclude that the entanglement strength as assessed by the trace negativity is always greater for our non-Gaussian states even though they exhibit the same entanglement strength as evaluated using the inseparability criterion. Next we examine an operational consequence of this fact.

VI. TWO EXAMPLES

With the knowledge from the previous section, we are able so show how these results apply to two practial situations where one may use this non-Gaussian state. The examples we will give involve using the non-Gaussian state as the entanglement for continuous variable quantum teleportation and the entanglement cloner attack in coherent state quantum key distribution.

A. Teleportation

It has previously been noted that pure non-Gaussian states, if used as a resource in CV teleportation, can teleport coherent states with a higher fidelity than Gaussian resource states with the same inseparability criterion [17]. We now demonstrate that this unusual property is also possessed by our mixed non-Gaussian states.

The fidelity of continuous variable teleportation performed using our non-Gaussian state

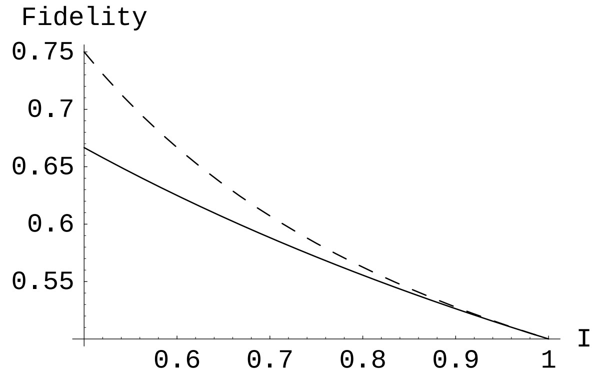

FIG. 2: Plot of the pure state fidelity as per Eq. (17) (solid line) and mixed state fidelity as per Eq. (16) (dashed line) for $p = 0.5$ as a function of r. Note that as the squeezing increases I decreases.

as the entanglement resource is simply given by the weighted sum of the fidelities achieved by its two constituent Gaussian states,

$$\mathscr{F}_{\hat{\rho}_{rm1}} = \frac{p}{1+e^{-2r}} + \frac{1-p}{2}. \tag{16}$$

The fidelity of CV teleportation when using Gaussian state with squeeze parameter given by Eq. (12) is

$$\frac{1}{1+e^{-2s}} = \frac{1}{1+I} \tag{17}$$

and rewriting Eq. (16) using Eq. (12) the non-Gaussian state fidelity is

$$\frac{p}{1+e^{-2r}} + \frac{1-p}{2} = \frac{p^2}{I+2p-1} + \frac{1-p}{2}. \tag{18}$$

Fig. 2 shows this fidelity as a function of I compared to the fidelity given by Eq. (16) when $p = 0.5$. The fidelity for the mixed state is higher for all values of r. At this point we now claim that this feature, that the non-Gaussian resource state gives higher fidelity, holds for all p and all r as long as the inseparability comparison is satisfied. To show this, first write down the difference of the mixed state fidelity (Eq. (18)) from the pure state fidelity (Eq. (17)) with the same inseparability criterion

$$\frac{p^2}{I+2p-1} + \frac{1-p}{2} - \frac{1}{1+I} = \frac{(I-1)^2(1-p)}{2(I+1)(I-1+2p)}. \tag{19}$$

Note that as $0 \leq p \leq 1$ and $1-p < I \leq 1$ the term on the right hand side is non-negative. Hence it follows that

$$\frac{p^2}{I+2p-1} + \frac{1-p}{2} \geq \frac{1}{1+I}. \tag{20}$$

The difference between the two fidelities is maximised when $I = 1 - p$ (i.e. $e^{-2r} = 0$, infinite squeezing in the mixed state) and $p = 2 - \sqrt{2}$. The maximum difference in Fidelity is $\frac{1}{2}\left(3 - 2\sqrt{2}\right) \approx 8.6\%$.

B. Continuous Variable QKD

The optimum individual eavesdropper attack on continuous variable, coherent state, quantum key distribution has been shown to be via an entangling cloner [7]. One might think that because of the increased entanglement of the non-Gaussian source, that it would be a superior resource for implementing such a cloner. On the other hand Grosshans [18] has shown, in a general way, that Gaussian attacks are optimal. Our mixed non-Gaussian source offers an accessible example of Grosshans' general result.

The quantity which determines the rate at which the two main parties can form a key via the reverse reconsiliation protocol is given by the difference between their mutual information and the mutual information between an evesdropper and the recieving station [18]. That is

$$\Delta I = I(B; A) - I(B; E) \tag{21}$$

where $I(B; A) = H(Q_B) - H(Q_B|Q_A)$ and $H(\cdot)$ is the entropy of the measured data Q and $H(\cdot|\cdot)$ is the mutual entropy between the two parties. The mutual information for the protocol described in [7] can be written as

$$\Delta I(A, \eta, N) = \frac{1}{2}\log_2\left(\frac{\left(\frac{\eta}{A} + (1-\eta)N\right)^{-1}}{\eta + (1-\eta)N}\right) \tag{22}$$

where η is the effiency of the channel and Alice modulates the input with variance A and $(1-\eta)N$ is the variance of the noise added by the the channel due to thermal effects.

If now one assumes that an eavesdropper has full control over the channel, one must make the conservative assumtion that she can control and manipulated all aspects of the channel. Here we have a channel with a given loss rate η, which has presumably been characterised by Alice and Bob. Eve now has access to the input state to the loss modes of the channel. Presumably, Alice and Bob would notice if the loss modes contained a state significantly different from a thermal state when they characterise the channel. So the eavesdropper must try to maintain a state which has similar properties to the thermal state that Alice and Bob are expecting.

Here we are going to assume that Alice and Bob only characterise their channel by computing the first and second order moments of the data they collect from the channel. They do this because they are confident that the channel is Gaussian. The eavesdropper may try to exploit this by choosing to use her entangling cloner to attach a mixed state of the form in equation 9 such that

$$N = pN_p + (1 - p) \tag{23}$$

where N is the variance of the state entering the loss mode of the channel, N_p is the single mode variance from the non-vaccum part of the state in equation 9 and p is as per the same equation.

Given this restriction on the eavesdroppers state which will decieve Alice and Bob and using the knowledge that Alice and Bob don't have about the true nature of the channel, it is possible to calculate the true difference in mutual information between Alice and Bob and Eve and Bob as

$$\Delta I_{mix}(A, \eta, N, N_p) = \frac{N-1}{N_p-1}\Delta I(A, \eta, N_p) + (1 - \frac{N-1}{N_p-1})\Delta I(A, \eta, 1). \tag{24}$$

The eavesdropper's objectives is to minimise the mutual information between Alice and Bob and maximise that between herself and Bob, in effect reducing ΔI. She will achieve this when

$$\Delta I(A, \eta, N) - \Delta I_{mix}(A, \eta, N, N_p) > 0. \tag{25}$$

However, an exhaustive numerical search reveals no parameters A,η,N and N_P satisfying $A, N, N_p > 1$,$0 < \eta < 1$ and $N_p > N$ which solve this inequality. A particular example of this is shown in Figure 3.

Hence even though the mixed non-Gaussian entanglement has higher entanglement and can produce higher fidelity in continious variable teleportation than the corresponding pure Gaussian entanglement, its use in an entangleing cloner attack on a continouous variables QKD protocol does not actually achieve better eavesdropping.

VII. DISCRETE VERSUS CONTINUOUS LOSS

An objection that could be raised regarding our comparison technique is that we are comparing states that in general have different photon numbers and thus have different

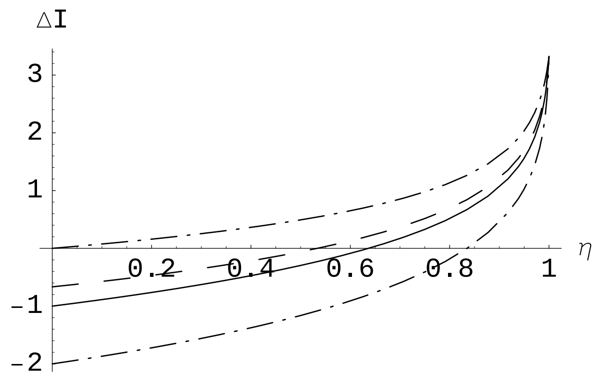

FIG. 3: A plot of the different of information rates between Alice and Bob and Eve and Bob as a function of channel loss. The solid line is where an eavesdropper uses a gaussian state in her entangling cloner. The dashed line is where she uses a mixed state in equation 9. The dot dashed lines are the rates that would be achieved if only the components of the mixed state were used. The vaccum state is the top dot-dashed line and the squeezed state is the lower dot-dashed line. Since these two states are mixed to produce a state with the same first and second order moments as that used in the solid line (see text), the resultant differerence in information rate given by the dashed line only disadvantages Eve in her attack.

entanglement per photon. It could then be argued that a fairer comparison would be to keep both I and the average photon number the same. We will now show that this scenario leads to identical conclusions to those before.

Consider the following situation. Greg and Nancy both start with identical two-mode squeezed states. Both reduce the average photon numbers in their states by the same ratio but via different processes. Greg subjects his beams to equal levels of continuous loss, such as is induced by passage through beamsplitters. Nancy instead subjects her beams to discrete loss by randomly either completely blocking or completely transmitting both her beams. If the transmission efficiency of Greg's state through the continuous loss, η is equal to the probability that Nancy's state is allowed to be transmitted, p, then the average photon numbers of the the two states after the loss processes are equal. Moreover we will now show that I is also equal for the two states. However, Greg's state has remained Gaussian whilst Nancy's state has become non-Gaussian.

Greg's state after the continuous loss is a two mode Gaussian state with mean zero and covariance matrix given by

$$\begin{pmatrix} \eta\cosh(2r)+1-\eta & 0 & -\eta\sinh(2r) & 0 \\ 0 & \eta\cosh(2r)+1-\eta & 0 & \eta\sinh(2r) \\ -\eta\sinh(2r) & 0 & \eta\cosh(2r)+1-\eta & 0 \\ 0 & \eta\sinh(2r) & 0 & \eta\cosh(2r)+1-\eta \end{pmatrix}. \tag{26}$$

Note that this covariance matrix is idential to that of the state given in Eq. (9) when $\eta = p$. In other words it has the same covariance matrix as Nancy's state. Because of this the inseparability criterion is the same for Greg and Nancy's states. The negativity for Greg's state is given by

$$\frac{(1+\eta(e^{-2r}-1))^{-1}-1}{2}. \tag{27}$$

Using Eq. (12) to rewrite this in terms of the inseparabiltiy criterion and $\eta = p$ gives the negativity to be

$$\frac{I^{-1}-1}{2}, \tag{28}$$

which is the same as the equation for the pure Gaussian state negativity, Eq. (14). So the comparison between Greg and Nancy's state negativities is identical to that already plotted in Fig. 1, i.e. the non-Gaussian state resulting from discrete loss is always more entangled.

VIII. CONCLUSION

In this paper we have discussed the non-Gaussian mixed state formed from the random mixture of a two mode squeezed vaccum and a standard vaccum state. We have shown that this state exhibits higher entanglement than an equivalent Gaussian state, where by equivalent we mean that both exhibit the same level of second order correlation as measured by the inseparability criterion. From a more operational point of view we have shown that our non-Gaussian state, if used as a resource state for continuous variable teleportation, can achieve fidelities superior to that of an equivalent Gaussian resource state. Such properties have previously been observed for pure non-Gaussian entangled states. One might have speculated that these were "deep" quantum effects, arising from Wigner function negativity. Our mixed non-Gaussian states do not exhibit Wigner function negativity thus negating this speculation.

We have used our non-Gaussian state as an example to illustrate the optimality of Gaussian attacks in continuous variable quantum key distribution. We showed explicitly that

even though the non-Gaussian state is more entangled, and even if Alice and Bob blindly assume that the attack is Gaussian, it does not lead to an advantage for Eve. We have also used our results to study the very distinct characteristics of discrete and continuous loss on entanglement.

Although not trivial, the production of our suggested states are within current technological capabilities.

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